Numerical Differentiation

Using Newton's forward interpolation :

$$f(x) \simeq y_0 + u \Delta y_0 + \frac{u(u-1)}{2!} \Delta^2 y_0$$

$$+ \frac{u(u-1)(u-2)}{3!} \Delta^3 y_0$$

$$+ \frac{u(u-1)(u-2)(u-3)}{4!} \Delta^4 y_0 + \cdot$$

Now we can differentiate to achieve

$f'(x)$, $f''(x)$, so on. $\quad u = \frac{x-x_0}{h} \Rightarrow \frac{du}{dx} = \frac{1}{h}$

$$f'(x) \approx \frac{1}{h}\left[\Delta y_0 + \frac{2u-1}{2!}\Delta^2 y_0 + \frac{3u^2-6u+2}{3!}\Delta^3 y_0 + \frac{4u^3-18u^2+22u-6}{4!}\Delta^4 y_0 + \cdots\right.$$

Do it yourself

$$f''(x) \approx \frac{1}{h^2}\left[\Delta^2 y_0 + \frac{6u-6}{3!}\Delta^3 y_0 + \frac{12u^2-36u+22}{4!}\Delta^4 y_0 + \cdots\right.$$

Using Newton's Backward Interpolation

$$f(x) \approx y_n + u \Delta y_{n-1} + \frac{u(u+1)}{2!} \Delta^2 y_{n-2} + \frac{u(u+1)(u+2)}{3!} \Delta^3 y_{n-3} + \cdots$$

Differentiate to achieve $f'(x)$, $f''(x)$ and so on.

$f'(x) \simeq$

$f''(x) \simeq$

Using Gauss Forward Interpolation

H.W.

$$f'(x) \simeq \frac{1}{h}\left[\Delta y_0 + \frac{2u-1}{2!}\Delta^2 y_{-1} + \frac{3u^2-1}{3!}\Delta^3 y_{-1} + \frac{4u^3-6u^2-2u+2}{4!}\Delta^4 y_{-2} + \cdots\right.$$

Using Gauss Backward Interpolation

H.W.

$$f'(x) \simeq \frac{1}{h}\left[\Delta y_{-1} + \frac{2u+1}{2!}\Delta^2 y_{-1} + \frac{3u^2-1}{3!}\Delta^3 y_{-2} + \frac{4u^3+6u^2-2u-2}{4!}\Delta^4 y_{-2} + \cdots\right.$$

Ex (H.W.)

Find the value of $f'(1)$, $f''(1)$, $f'(2)$, $f''(2)$, $f'(4)$ from the following values:

x	1	2	4	8	10
y	0	1	8	27	34

Numerical Integration

Sometimes using the analytical methods of integration is impossible to evaluate a definite integral

eg $$\int_0^{\pi/2} \sqrt{\sin x}\, dx$$

$$\int_0^1 e^{x^2} dx, \quad \int_2^3 \frac{e^x}{x} dx$$

We may use numerical integration for that.

Error of approximation

Suppose we wish to evaluate the definite integral $I = \int_a^b f(x)\,dx$.

1st step: Approximate $f(x)$ by a polynomial $\phi(x)$ of suitable degree

2nd Step integrate $\int_a^b f(x)\,dx \approx \int_a^b \phi(x)\,dx$

The difference $\left[\int_a^b f(x)\,dx - \int_a^b \phi(x)\,dx\right]$ = error of approximation

General Quadrature formula

Suppose $f(x)$ is unknown except for some given equidistant x values, say $x_0, x_0+h, x_0+2h, \ldots, x_0+nh$

We divide $[a, b]$ into n sub-intervals of width h.

$a = x_0, \; x_1 = x_0 + h, \; x_2 = x_0 + 2h, \ldots$ so on

$$I = \int_{x_0}^{x_0 + nh} f(x)\, dx$$

put $\dfrac{x - x_0}{h} = u \Rightarrow dx = h\,du$

$$I = h \int_0^n f(x_0 + hu)\, dx$$

$$= h \int_0^n E^u f(x_0)\, dx$$

$$= h \int_0^n (1+\Delta)^u f(x_0)\, dx$$

$$= h \left[\int_0^n \left[1 + u\Delta + \binom{u}{2}\Delta^2 + \binom{u}{3}\Delta^3 + \cdots \right] f(x_0)\, dx \right.$$

$$= h \left[n f(x_0) + \frac{n^2}{2} \Delta f(x_0) + \frac{\left(n^3/3 - n^2/2\right)}{2} \Delta^2 f(x_0) + \cdots \right.$$

$$I = h\left[n y_0 + \frac{n^2}{2}\Delta y_0 + \frac{\frac{n^3}{3} - \frac{n^2}{2}}{2}\Delta^2 y_0 + \frac{\frac{n^4}{4} - n^3 + n^2}{6}\Delta^3 y_0 + \ldots \right]$$

General Gauss Legendre Quadrature formula (for equidistant coordinate)